A Detailed Analysis of the Ant Colony Optimization Enhanced Particle Filters

J.P. Zhong[1] , Y.F. Fung[2]

[1] Department of Computer Science, University of Hamburg
Vogt-Koelln-Str. 30., 22527
Hamburg, Germany
zhong@informatik.uni-hamburg.de
[2]Department of Electrical Engineering, The Hong Kong Polytechnic University
Hung Hom, Kowloon, Hong Kong
eeyffung@inet.polyu.edu.hk

Abstract. Particle filters, as a kind of non-linear/non-Gaussian estimation method, are suffered from two problems when applied to cases with many states dimensions, namely particle impoverishment and sample size dependency. Previous papers from the authors have proposed a novel particle filtering algorithm that incorporates Ant Colony Optimization (PF-ACO), to alleviate effect these problems. In this paper, we will provide a theoretical foundation of this new algorithm. A theorem that validates the PF-ACO introduces a smaller Kullback-Leibler Divergence (K-L divergence) between the proposal distribution and the optimal one when comparing to those produced by the generic PF is discussed.

Keywords: Ant Colony Optimization, Combinatorial Optimization, Metaheuristic Methods, Nonlinear Estimation, Particle Filters

1 Introduction

Particle Filter (PF) is based on point mass particles that represent the probability densities of the solution space and it is widely used for solving non-linear and non-Gaussian state estimation problems[1]. As an alternative method of Kalman Filter [2, 3], it is widely used in applications under non-linear and non-Gaussian environments, The advantage of PF is that it can estimate any probability distribution [4] with an infinite number of samples. Although this optimal estimation is not available in real applications, it can still produce better results in the non-linear/non-Gaussian environment. However, particle impoverishment is inevitably induced due to the random particles prediction and re-sampling applied in generic PF [5], especially for problems that come with a huge number of state dimensions. After a number of iterations, if the generated particles are too far away from the likelihood distribution, their particle weights will approach zero and only a few particles are left which have significant weights, making other particles not efficient to produce accurate estimate estimation results.

Therefore, there are other enhanced PF algorithms that employ different sampling strategies to minimize the impoverishment effect and these strategies include Binary

Search[6], Systematic Resampling[7] and Residual Resampling[8], whose target are copying the important samples and discarding insignificant ones by different calculation and selection methods mainly based on their weights. However, at the meantime, the robustness of the filter is lost, because the diversity of particles is reduced by a certain extent [9]. In [10, 11], a metaheuristic method is introduced, in which the Ant Colony Optimization (ACO) is applied to optimize the particle distribution, which will be introduced in the next section. Validating the effectiveness of the PFACO based on the K-L divergence is included in Section 3 while discussion followed by conclusions, are presented in Section 4 and 5 respectively.

2 Particle Filters

2.1 Generic Particle Filters

Particle filters are algorithms to perform recursive Bayesian estimation using Monte Carlo simulation and importance sampling, in which the posterior density is approximated by the relative density (weights) of particles observed in the state space. The posterior can be approximated by the weighted summation of every particle as follows:

$$p(x_{0:k} \mid y_{1:k}) \approx \sum_{i=1}^{N} w_k^i \delta(x_{0:k} - x_{0:k}^i) \tag{1}$$

where the weighting value of particle i at time-step k, w_k^i is updated according to Eq 2.

$$w_k^i \propto w_{k-1}^i \frac{p(y_k \mid x_k^i)\, p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, y_k)} \tag{2}$$

It can be shown that as $N \to \infty$ the approximation (Eq. 1) approaches the true posterior density $p(x_k \mid y_{1:k})$ [12].

However, in problems that involve a huge number of dimensions, such as the multi-robot SLAM problem, a large number of particles must be included in order to maintain an accurate estimation, the generic resampling method is not sufficient to avoid the impoverishment and size dependence problems. Consequently these problems will become very severe after a number of iterations, rendering a large portion of the particles negligible and reducing the accuracy of the estimation results.

2.2 Ant Colony Optimization enhanced PF

In order to optimize the re-sampling step of the generic particle filter, we incorporate ACO into the PF and utilize the ACO before the updating step [10, 11]. A single ant will replace the particle and they will move based on the choice of possible routes towards the local peak of the optimal proposal distribution function.

The parameter $\tau(t)$, as shown in Eq. 3, is affected by every movement of the particle by the following equation:

$$\begin{cases} \tau_{i*}(t+1) = (1-\rho)\tau_{i*}(t) + \Delta\tau_{i*}(t) & \text{set of particles lie in the movement path} \\ \tau_{i*}(t+1) = (1-\rho)\tau_{i*}(t) & \text{set of other particles} \end{cases} \tag{3}$$

where $0 < \rho \leq 1$ is the pheromone evaporation rate, $\Delta\tau$ is a constant enhanced value if particle j is located between the starting particle and the end point.

The heuristic function (β) is defined as the reciprocal of the distance between two particles (end points):

$$\eta_{i*}(t) = \frac{1}{d_{i*}} \tag{4}$$

Finally, the optimization step runs iteratively based on a probability function obtained from Eq 5. It represents the probability of a particle i selecting particle j among N-1 particles as the moving direction.

$$p_{ij}(t) = \frac{[\tau_{ij}(t)]_{\alpha}[\eta_{ij}(t)]_{\beta}}{\sum_{s \in \text{all particles}} [\tau_{is}(t)]_{\alpha}[\eta_{is}(t)]_{\beta}} \tag{5}$$

The initial value of parameter α (τ_{i*}) equal to the particle weight, as stated in Eq. 6.

$$\tau_{i*}(0) = w_{*} \tag{6}$$

When the ACO algorithm converges and P_{ij} approaches 1[13], it implies that the particle i re-locates at a closer proximity of particle j. A pseudo-program describing the PF-ACO algorithm is given below.

Algorithm: The PF_{ACO} Algorithm

$[\{x_k^i, w_k^i\}_{i=1}^N] = PFACO[\{x_{k-1}^i, w_{k-1}^i\}_{i=1}^N, y_k]$

(1) The initialization and prediction steps (these are same as the original PF algorithm)

(2) ACO enhanced PF

While the distance between particles' measurement and the true measurement are not within a certain threshold and the iteration number does not exceed the maximum value

- Choose particle i whose distance is within the threshold
- Select the moving target based on the probability (Eq. 5)
- Move towards the target with a constant velocity
  Update the parameters of the ACO (e.g. η, τ), and particle weights

End While

(3) Update Step & Resampling

---

3 Theoretical Foundation of PF-ACO

In this section, a theorem will be proposed together with its proof in order to elaborate how the PF_{ACO} can produce better solution when compared to the generic PF, which employs a transition function as the proposal distribution.

Theorem: With the convergence nature of ACO, the PF_{ACO} can always achieve the optimal proposal distribution when the ACO converges to an optimal solution.

Proof: In its generic form, a transition model is often employed as the predicted proposal distribution:

$$q(x_k \mid x_{k-1}, y_k) = p(x_k \mid x_{k-1})_{tran} \tag{7}$$

while the optimal distribution is defined by Eq. 8.

$$q(x_k \mid x_{k-1}^i, y_k)_{opt} = p(x_k \mid x_{k-1}^i, y_k) \tag{8}$$

where the $q(x_k \mid x_{k-1}, y_k)$ in Eq. 8 represents the true distribution of the likelihood of state *x* with all previous states and observations are given. Since the probability is difficult to be integral, so we usually employ the transition function $p(x_k \mid x_{k-1})_{tran}$ to approximate the true distribution.

The second term $p(x_k \mid x_{k-1}^i, y_k)$ in an application represents the probability that moving to state x_k in time *k*, given the samples in previous time step x_{k-1} and the measurement y_k.

In ideal cases, the proposal distribution should consider two kinds of noises: noises from the odometer and noise from the sensor. However, the generic transition model only incorporates the probability from motion detector noise. Consequently, the generic transition model can approximately equivalent to the optimal model only if either of following two conditions is satisfied.

1) The odometer has no error in measurement, or
2) the odometer noise has similar noise variance as the observation sensor.

Nevertheless, the above two conditions are difficult to achieve in most of our experiments due to the different variances contributed by various sensors' measurement errors. The observation sensors, such as laser and vision sensors, are getting more accurate, but this is not the case for an odometer. With different magnitude of variance levels, traditional transition model based on the odometer is not as suitable as it used to be, especially in experiments that include observation sensors and motion sensors.

In order to prove ACO is able to solve this problem, Kullback–Leibler divergence (K-L divergence) is introduced. K-L divergence is a non-symmetric measure of the difference between two probability distributions. The approximation of K-L divergence [14] is generated by a set of sample data set: $s_1, s_2, \ldots s_N$, based on the model density *p(x)*, so

$$D(p \| q) \approx \frac{1}{N}\sum_{i=1}^{N}[\log p(x(n)) - \log q(x(n))] \tag{9}$$

For the generic PF, the above K-L Divergence equals to

$$D(p \| q) \approx \frac{1}{N}\sum_{i=1}^{N}[\log p(x_k(n) \mid x_{k-1}(i), y_k) - \log q(x_k(n) \mid x_{k-1}(i))] \tag{10}$$

To evaluate the K-L Divergence, we take *N* Monte Carlo samples in state space for x_k, and calculate their probability density given the condition of particle $x_{k-1}(i)$ and y_k.

Based on Eq. 10, it is trivial to derive that the ACO algorithm converges if and only if $p_{ij}(k) = 1$, which indicates the necessary and sufficient conditions of ACO convergence is $d_{ij} = 0$ or $\lim_{t\to\infty} \eta_{ij}(k) \to \sum_{s\in \text{all particles}} \eta_{is}(k)$. Thus, **majority of particles will be located around the peak of the mixture likelihood density function.**

Secondly, assuming that samples $\hat{x}_1, \hat{x}_2, ..., \hat{x}_n$ in the optimal proposal distribution are taken, in order to approach the optimal proposal distribution according to the definition of K-L Divergence and our Theorem 1, we will derive the relationship between the number of samples and the optimal distribution. If it is necessary to have *M* samples $(\tilde{x}_k, \tilde{x}_{k+1}, ..., \tilde{x}_{k+M})$ in order to generate *N* samples $(\tilde{x}_k, \tilde{x}_{k+1}, ..., \tilde{x}_{k+N})$ in the continuous optimal proposal distribution, the number of samples needed to be considered is proportional to the second derivative of the optimal distribution according to the interpolation error[15], which can be illustrated by Fig. 1 and Eq. 11.

$$\begin{aligned} M &= \lambda N[f''(s_k) + f''(s_{k+1}) + \ldots + f''(s_{k+N})]/N \\ &= \lambda \bullet [f''(s_k) + f''(s_{k+1}) + \ldots + f''(s_{k+N})] \end{aligned} \tag{11}$$

In Eq. 11, λ is a constant, indicating that **number of *M* is proportional to the summation of the second derivatives of all samples in this interval.**

As shown in Fig. 1, *k* samples in the optimal Gaussian distribution are taken in uniform intervals, and within which, *M* samples are included in the original discrete distribution, that is, $M_{i_k} \in \{M_1, M_2, \ldots, M_k\}$. Similarly, samples in the proposal distribution are also separated into k intervals, that is $N_{i_k} \in \{N_1, N_2, \ldots, N_k\}$. Given the convergence of Ant Colony Optimization algorithm [13], if a certain continuous optimal proposal distribution are divided into *M* samples, the sample ${s^+}_t$ move closer to these *M* samples after the ACO improvement.

Therefore, we can compare two K-L Divergence before and after the ACO improvement, from definition of Eq. 9 and 10,

$$D(p \| q) \approx \frac{1}{M}\sum_{i_k=1}^{k}\sum_{n\in \text{interval k}}[\log \hat{p}(x_k(n) \mid x_{k-1}(i), y_k) - \log \tilde{q}(x_k(n) \mid x_{k-1}(i))] \tag{12}$$

and

$$D(p \| q^{+}) \approx \frac{1}{M} \sum_{i_k=1}^{k} \sum_{n \in \text{interval k}} [\log \hat{p}(x_k(n) \mid x_{k-1}(i), y_k) - \log \tilde{q}^{+}(x_k(n) \mid x_{k-1}(i))] \tag{13}$$

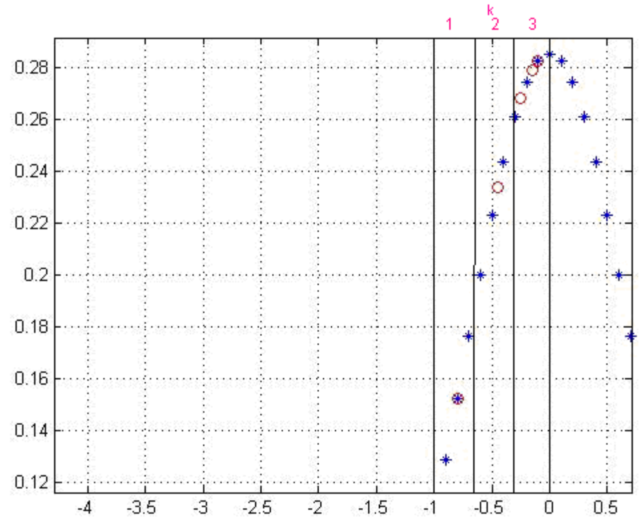

Fig.1 Demonstration of the samples size M and N.
Within [-1, 0], there are k=3 intervals. In the 1st and 2nd interval, M_1=1 and M_2=1 samples may be sufficient to represent the distribution because all the second derivatives in this interval are nearly equal to zero ,and M_3=3 sample are needed to re-construct the distribution.

Let the sequence $\hat{M}_{i_k} \in \{\hat{M}_1, \hat{M}_2, \ldots, \hat{M}_k\}$ denote the required particle number in each interval based on Equation 18. After sufficient iterations to achieve the optimal solution, if in an interval that the required particle number $\hat{M}_{i_k} \le N_{i_k}$, such as k = 1,2 in Fig 1, it is trivial that

$$D(p \| q) = D(p \| q^{+}). \tag{14}$$

If within the intervals that the required particle number $\hat{M}_{i_k} > N_{i_k}$, such as k = 3 as illustrated in Fig 1, then

$$\begin{aligned} D(p \| q) - D(p \| q^{+}) &\approx \frac{1}{M} \sum_{n=1}^{M} [\log \frac{p(\hat{x}_k(n) \mid x_{k-1}(i), y_k)}{q(\tilde{x}_k(n) \mid x_{k-1}(i))} - \log \frac{p(\hat{x}_k(n) \mid x_{k-1}(i), y_k)}{q(\tilde{x}_k^{+}(n) \mid x_{k-1}(i))})] \\ &= \frac{1}{M} \sum_{n=1}^{M} [\log \frac{p(\hat{x}_k(n) \mid x_{k-1}(i), y_k)}{q(\tilde{x}_k(n) \mid x_{k-1}(i))} - \log \frac{p(\hat{x}_k(n) \mid x_{k-1}(i), y_k)}{p(\hat{x}_k(n) + \varepsilon \mid x_{k-1}(i), y_k)}] \\ &\to \frac{1}{M} \sum_{n=1}^{M} [\log \frac{p(\hat{x}_k(n) \mid x_{k-1}(i), y_k)}{q(\tilde{x}_k(n) \mid x_{k-1}(i))} - 0] \\ &> 0 \end{aligned} \tag{15}$$

The above convergence comes from the convergence of Ant Colony Optimization [13]. So within the intervals that the required particle number $\hat{M}_{i_k} > N_{i_k}$, we get

$$D(p \| q) > D(p \| q^{+}) \tag{16}$$

Given a small number ε, with sufficient iterations, we can always achieve arbitrarily small K-L Divergence. Therefore, when we take summation in all intervals for K-L Divergence calculation, we can conclude that $D(p \| q) > D(p \| q^{+})$. □

4 Discussion

The above theorem qualitatively shows that the proposal distribution can ultimately achieve the optimal solution with Ant Colony Optimization.

From the proof presented in Section 3, with reference from the formulation of the combinatorial optimization problem framework, the optimal proposal distribution problem being considered can be classified as a combinatorial optimization problem satisfying Eq. 17.

$$Min(D(q(s_t(i) \mid s_{t-1}(i), z), p(s_t \mid s_{t-1}, z))$$
$$s.t. \begin{cases} \sum_{i=1}^{M} w(i) = 1 \\ w_t(i) = \mu \dfrac{p(z_t \mid s_t(i)) p(s_t(i) \mid s_{t-1}(i))}{q(s_t(i) \mid s_{t-1}(i), z_t)} \end{cases} \tag{17}$$

where *D()* is the K-L divergence between two distributions.

Because the model is not known in advance in the problem, a heuristic method is considered to be one of the possible solutions in this paper. Directly speaking, we know that one important factor of tuning the proposal distribution, so that any similar metaheuristic can also be applied to solve this problem.

5 Conclusions

As a continuous study of the Ant Colony Improved Particle Filter (PF_{ACO}), a theoretical deduction of the improvement process is included in this paper. Our theorem validates that the PF_{ACO} optimizes the proposal distribution to generate a smaller Kullback–Leibler divergence value than that obtained from generic PF. From the theorem, we further discuss a framework to optimize this particle distribution based on combinatorial optimization. Using this framework, metaheurstic methods, e.g. ACO, or other methods can be applied to introduce better estimation results in non-linear as well as non-Gaussian engineering estimation problems.

Acknowledgments. The work presented in this paper is supported by the Department of Electrical Engineering of the Hong Kong Polytechnic University.

References

1. N.J.Gordon, D.J.Salmond, and A.F.M.Smith, "Novel approach to nonlinear/non-Gaussian Bayesian state estimation." IEE Proceedings F Rader and Signal Processing vol. 140 no. 2, pp. 107-113. 1993.
2. S.F.Schmidt, "The Kalman filter - Its recognition and development for aerospace applications," Journal of Guidance, Control, and Dynamics, vol. 4, no. 1. pp.4-7, 1981.
3. E.A.Wan and R.Van der Merwe, "The unscented Kalman filter for nonlinear estimation." Proceeding of IEEE Symposium of Adaptive Systems for Signal Processing, Communication, and Control , pp. 153-158. 2000.
4. D.Crisan and A.Doucet, "A survey of convergence results on particle filtering methods for practitioners," IEEE Transactions on Signal Processing, vol. 50, no. 3. pp.736-746, 2002.
5. M.G.S.Bruno and A.Pavlov, "Improved particle filters for ballistic target tracking." IEEE International Conference on Acoustics, Speech, and Signal Processing vol. 2, pp. 705-708. 2004.
6. A.Doucet, S.Godsill, and C.Andrieu, "On sequential Monte Carlo sampling methods for Bayesian filtering," Statistics and Computing, vol. 10, no. 3. pp.197-208, 2000.
7. M.Isard and A.Blake, "CONDENSATION: Conditional Density Propagation for Visual Tracking," International journal of computer vision, vol. 29, no. 1. pp.5-28, 1998.
8. J.U.Cho, S.H.Jin, X.D.Pham et al., "A Real-Time Object Tracking System Using a Particle Filter." 2006 IEEE/RSJ International Conference on Intelligent Robots and Systems. pp. 2822-2827. 2006.
9. C.Stachniss, G.Grisetti, and W.Burgard, "Recovering particle diversity in a Rao-Blackwellized Particle Filter for SLAM after actively closing loops." Proceedings of the 2005 IEEE International Conference on Robotics and Automation , pp. 18-22. 2005.
10. J. Zhong, Y. Fung, and M. Dai, "A biologically inspired improvement strategy for particle filter: Ant colony optimization assisted particle filter," International Journal of Control, Automation and Systems, vol. 8, no. 3. pp.519-526, 2010.
11. J. P. Zhong and Y. F. Fung, "A biological inspired improvement strategy for Particle Filters." Industrial Technology, 2009.ICIT 2009.IEEE International Conference on , pp. 1-6. 2009. IEEE.
12. M.S.Arulampalam, S.Maskell, N.Gordon et al., "A tutorial on particle filters for online nonlinear/non-GaussianBayesian tracking," IEEE Transactions on Signal Processing, vol. 50, no. 2. pp.174-188, 2002.
13. T. Stutzle and M. Dorigo, "A short convergence proof for a class of ant colony optimization algorithms," IEEE Transactions on Evolutionary Computation, vol. 6, no. 4. pp.358-365, 2002.
14. J. R. Hershey and P. A. Olsen, "Approximating the Kullback Leibler divergence between Gaussian mixture models." Acoustics, Speech and Signal Processing, 2007.ICASSP 2007.IEEE International Conference on vol. 4, p.IV-317. 2007. Ieee.
15. P. J. Davis, Interpolation and approximation, 1975.